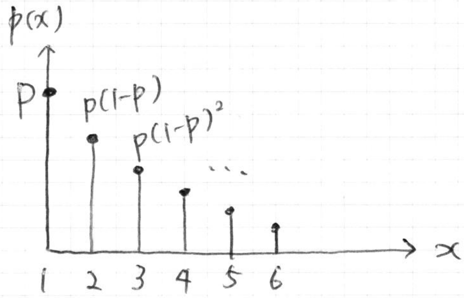
熱・統計力学演習〈第2回〉

$\boxed{1}$ (a) $p(x) = (1-p)^{x-1} \cdot p \quad (x = 1, 2, 3, \cdots)$



平均 $E(x) = \sum_{x=1}^{\infty} x \cdot p(x) = \sum_{x=1}^{\infty} x(1-p)^{x-1} \cdot p$

$= p\{1 + 2(1-p) + 3(1-p)^2 + \cdots\}$

ここで $S_N = 1 + 2(1-p) + 3(1-p)^2 + \cdots$

$-)\ (1-p)S_N = \quad (1-p) + 2(1-p)^2 + \cdots$

$pS_N = 1 + (1-p) + (1-p)^2 + \cdots = \dfrac{1-(1-p)^N}{1-(1-p)}$

$1-p < 1$ なので、$N \to \infty$ で $pS_N = \dfrac{1}{p}$

$\therefore\ E(x) = 1/p$

分散 $V(x) = \sum_{x=1}^{\infty} (x - \frac{1}{p})^2 \cdot p(x) = \sum_{x=1}^{\infty} (x - \frac{1}{p})^2 \cdot (1-p)^{x-1} \cdot p$

$= p\{(1-\frac{1}{p})^2 + (2-\frac{1}{p})^2(1-p) + (3-\frac{1}{p})^2(1-p)^2 + \cdots\}$

ここで $S_N = (1-\frac{1}{p})^2 + (2-\frac{1}{p})^2(1-p) + \cdots$

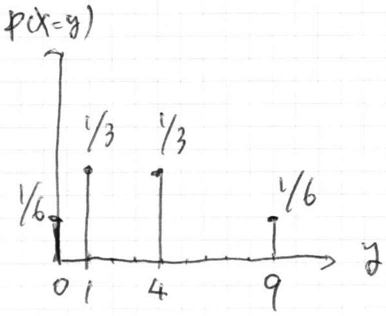
$-)\ (1-p)S_N = \quad (1-\frac{1}{p})^2(1-p) + \cdots$

$pS_N =$

さらに、$-)\ (1-p)S_N =$

$\therefore\ V(X) = (1-p)/p^2$

1 (b) $P(y=0)=1/6$, $P(y=1)=1/3$, $P(y=4)=1/3$, $P(y=9)=1/6$



$$E(X) = 0\cdot 1/6 + 1\cdot 1/3 + 4\cdot 1/3 + 9\cdot 1/6 = 19/6$$

$E(X^2)$

$$V(X) = 0^2\cdot 1/6 + 1^2\cdot 1/3 + 4^2\cdot 1/3 + 9^2\cdot 1/6 = 115/6 - (19/6)^2$$

$$= 329/36$$

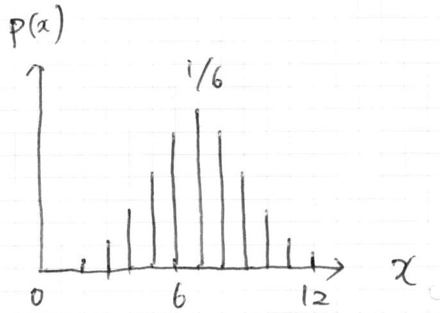
$690/36 - 361/36 =$

(c) $p(2) = 1/36$

$p(3) = 2/36$

$\vdots$

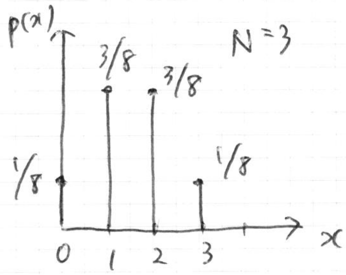
$p(12) = 1/36$



$E(X) = 7$, $V(X) = 35/6$

(d) $p(x) = \dfrac{N!}{x!\,(N-x)!}\left(\dfrac{1}{2}\right)^N$

$N=3$のとき: $p(x) = \dfrac{3!}{x!\,(3-x)!}\left(\dfrac{1}{2}\right)^3$



$E(X) = 3/2$, $V(X) = 3/4$

$\boxed{2}$ (a) $p(x) = \dfrac{\mu^x}{x!} e^{-\mu}$.

$$E(x) = \sum_{x=0}^{\infty} x\, p(x)\, dx = \left(0 \cdot 1 + 1 \cdot \frac{\mu}{1!} + 2 \cdot \frac{\mu^2}{2!} + \cdots\right) \cdot e^{-\mu}$$

$$= \mu \left(1 + \mu + \frac{\mu^2}{2!} + \cdots\right) \cdot e^{-\mu}$$

$$= \mu \cdot e^{\mu} \cdot e^{-\mu}$$

$$= \mu .$$

$$V(x) = \sum_{x=0}^{\infty} (x-\mu)^2 \cdot p(x)$$

$$= \sum_{x=0}^{\infty} x^2\, p(x) - \left(\sum_{x=0}^{\infty} x\, p(x)\right)^2$$

$$\sum_{x=0}^{\infty} x^2 p(x) = \sum_{x=0}^{\infty} x(x-1) \frac{\mu^x}{x!} e^{-\mu} + \sum_{x=0}^{\infty} x \cdot \frac{\mu^x}{x!} e^{-\mu}$$

$$= \mu^2 \left(1 + \mu + \frac{\mu^2}{2!} + \cdots\right) e^{-\mu} + \mu \left(1 + \mu + \frac{\mu^2}{2!} + \cdots\right) \cdot$$

$$= \mu^2 + \mu .$$

$$V(x) = \mu^2 + \mu - \mu^2 = \mu$$

(b) $p(x) = \lambda e^{-\lambda x}$.

$$E(x) = \int_{-\infty}^{\infty} x\, p(x)\, dx = \int_0^{\infty} x \lambda e^{-\lambda x}\, dx$$

$$= \left[-x e^{-\lambda x}\right]_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\, dx$$

$$= 1/\lambda .$$

$$V(x) = \int_0^{\infty} \left(x - \frac{1}{\lambda}\right)^2 \lambda e^{-\lambda x}\, dx = \int x^2 \lambda e^{-\lambda x}\, dx - \left(\int x \lambda e^{-\lambda x}\, dx\right)^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

(c) $p(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$E(x) = \int_{-\infty}^{\infty} x \cdot p(x)\,dx = \int_{-\infty}^{\infty} \{(x-\mu)+\mu\} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{\infty} (x-\mu) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \int_{-\infty}^{\infty} \mu \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= 0 + \mu \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx}_{=1.}$$

$$= \mu$$

$$V(x) = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$u = \dfrac{x-\mu}{\sigma}$, $du = \dfrac{dx}{\sigma}$

$$= \sigma^2 \underline{\int_{-\infty}^{\infty} u^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot (-2) \cdot \int_{-\infty}^{\infty} \left(-\frac{u^2}{2}\right) e^{-\frac{u^2}{2}} du$$

$$= \frac{1}{\sqrt{2\pi}} \cdot (-2) \cdot \left(-\frac{1}{2}\right) \cdot \sqrt{2\pi} = 1.$$

$$= \sigma^2$$

$\boxed{3}$

(勝率) $= 1 - p(0)$ ← 1回も6のゾロ目が出ない確率.

$= 1 - \left(\dfrac{35}{36}\right)^{24} = 0.4914\cdots$　　分が悪い!

$\boxed{4}$ (a)

$$p(x) = \frac{N!}{x!\,(N-x)!}\, p^x\, (1-p)^{N-x}$$

(b) 2項定理

$$(a+b)^n = \sum_{x=0}^{n} \frac{n!}{x!\,(n-x)!}\, a^x b^{n-x} \quad \text{より。}$$

$$p(x) = (p + (1-p))^N = 1^N = 1.$$

(c)

$$E(x) = \sum_{x=0}^{N} x \cdot p(x)$$

$$= \sum_{x=0}^{N} x \cdot \frac{N!}{x!\,(N-x)!}\, p^x (1-p)^{N-x}$$

$$= \sum_{x=0}^{N} \frac{N!}{(x-1)!\,(N-x)!}\, p^x (1-p)^{N-x}$$

$$= Np \underbrace{\sum_{x=1}^{N} \frac{(N-1)!}{(x-1)!\,\{N-1-(x-1)\}!}\, p^{x-1}\, (1-p)^{\overset{N-1-(x-1)}{\overbrace{N-x}}}}_{=1.}$$

$$= Np$$

$$\sum_{x=0}^{N} x^2 p(x) = \sum_{x=0}^{N} \underbrace{\{x(x-1) + x\}}_{\text{置きかえ。}} \frac{N!}{x!\,(N-x)!}\, p^x (1-p)^{N-x}$$

$$= \sum_{x=0}^{N} x(x-1) \frac{N!}{x!\,(N-x)!}\, p^x (1-p)^{N-x} + \sum_{x=0}^{N} x \frac{N!}{x!\,(N-x)!}\, p^x (1-p)^{N-x}$$

$$= N(N-1)p^2 \sum_{x=2}^{N} \frac{(N-2)!}{(x-2)!\,(N-2-(x-2))!}\, p^{x-2} (1-p)^{N-2-(x-2)} + Np \sum_{x=1}^{N} \frac{(N-1)!}{(x-1)!\,(N-1-(x-1))!}\, p^{x-1} (1-p)^{N-1-}$$

$$= N(N-1)p^2 + Np.$$

従って、

$$V(x) = \sum x^2 p(x) - \left(\sum x p(x)\right)^2 = \cancel{N^2p^2} - Np^2 + Np - \cancel{N^2p^2}$$

$$= Np(1-p)$$

$$= Npq$$

(d)

$$p(x) = \frac{N!}{x!(N-x)!} p^x (1-p)^{N-x}$$

$$= \frac{\overbrace{N(N-1)\cdots(N-x+1)}^{x\text{個}}}{x!} \cdot p^x \cdot (1-p)^{N-x}$$

$\boxed{\text{平均値：}Np}$

$$= \frac{(Np)^x}{x!} \cdot \left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\cdots\left(1-\frac{x-1}{N}\right) \cdot \left(1-\frac{Np}{N}\right)^{N\left(1-\frac{x}{N}\right)}$$

$\frac{x}{N}$ は $N\to\infty$ で $\to 1$.

ここで $Np=\lambda$ とする。

$N\to\infty$ の極限で、 $\lim_{N\to\infty}\left(1-\frac{\lambda}{N}\right)^N = e^{-\lambda}$ より、

$$p(x) \xrightarrow[N\to\infty]{} \frac{\lambda^x}{x!} \cdot e^{-\lambda}$$

5 (a) $P(0, t+\Delta t) = (1-\nu\Delta t)P(0,t)$

$P(n, t+\Delta t) = (1-\nu\Delta t)P(n,t) + \nu\Delta t \cdot P(n-1,t)$.

(b) (a)より.

$$\frac{P(n, t+\Delta t) - P(n,t)}{\Delta t} = -\nu\{P(n,t) - P(n-1,t)\}$$ とできるので.

$\Delta t \to 0$で
$$\begin{cases} \dfrac{\partial P(n,t)}{\partial t} = -\nu\{P(n,t) - P(n-1,t)\} & (n \geq 1) \\ \dfrac{\partial P(0,t)}{\partial t} = -\nu \cdot P(0,t) \end{cases} \quad \cdots (*)$$

(c) (b)より.
$P(0,t) = Ae^{-\nu t}$, $P(0,0) = 1$ より $P(0,t) = e^{-\nu t}$.

$P(n,t) = g(n,t)P(0,t)$ とおくと,

(d) (*)の左辺 =
$$P(0,t)\partial_t g(n,t) + g(n,t)\partial_t P(0,t)$$
$$= P(0,t)\{\partial_t g(n,t) - \nu g(n,t)\}$$

(*)の右辺 $= -\nu P(0,t)\{g(n,t) - g(n-1,t)\}$.

$\Rightarrow \partial_t g(n,t) = \nu g(n-1,t) \quad \cdots (**)$

$g(n,t)$ を Taylor展開すると, $g(n,t) = \displaystyle\sum_{k=0}^{\infty} \frac{g^{(k)}(n,0)}{k!} t^k$ と表せる。

ここで, (**)より,
$$\begin{cases} \partial_t^{(k)} g(n,t) = \nu^k g(n-k,t) & (0 \leq k < n-1) \\ \partial_t^{(n)} g(n,t) = \nu^n g(0,t) = \nu^n & (k=n) \\ \partial_t^{(k)} g = 0 & (k>n) \end{cases}$$

従って, $g(n,t) = \nu^n t^n / n!$ $\quad \therefore P(n,t) = \dfrac{(\nu t)^n}{n!} e^{-\nu t}$ …ポアソン分布.